MATH 512, FALL 14 COMBINATORIAL SET THEORY WEEK 5

Forcing that preserves large cardinals

Theorem 1. *Suppose that in V, κ is measurable and $\mathbb{P}$ is a forcing notion of cardinality less than κ. Then if G is $\mathbb{P}$-generic, $V[G] \models$ "κ is measurable".*

Proof. Let $j : V \to M$ be an elementary embedding with critical point κ derived from some measure on κ. So, we have $\kappa < j(\kappa)$ and $M^\kappa \subset M$. Since $\mathbb{P}$ is a poset of size less than κ, we may assume that $\mathbb{P} \in V_\kappa$. (That is because one can always construct a poset in V_κ isomorphic to $\mathbb{P}$.) Then since $j \restriction \kappa = \mathrm{id}$, we also have that $j \restriction V_\kappa = \mathrm{id}$, and so $j(\mathbb{P}) = \mathbb{P}$.

Let G be $\mathbb{P}$-generic over V. Then since $M \subset V$ and $\mathbb{P} = j(\mathbb{P})$, G is also $\mathbb{P}$-generic over M. And if $\dot{a}$ is a $\mathbb{P}$-name in V, then $j(\dot{a})$ is a $\mathbb{P}$-name in M. So we can consider its interpretation with respect to G. For $a \in V[G]$, define

$$j'(a) = j(\dot{a})_G,$$

where $\dot{a}$ is some name for a.

Claim 2. *j' is well defined.*

Proof. Suppose that τ and σ are two $\mathbb{P}$-names in V for an element $a \in V[G]$. More precisely, $\sigma_G = \tau_G = a$. Then there is some $p \in G$, such that $p \Vdash \sigma = \tau$. Then $j(p) = p \Vdash j(\sigma) = j(\tau)$. Note that by elementarity, $j(\tau)$ and $j(\sigma)$ are $j(\mathbb{P}) = \mathbb{P}$-names in M. So, $M[G] \models j(\sigma)_G = j(\tau)_G$. Thus the map is well defined.

□

Then $j' : V[G] \to M[G]$ is an elementary embedding with critical point κ with $j' \in V[G]$. So, $V[G] \models$ "κ is measurable".

□

It turns out that the above theorem is true for any large cardinal property. Namely, we can replace "measurable" in the statement of the theorem with inaccessible, Mahlo, weakly compact, strongly compact, supercompact, etc.

Lifting Elementary Embeddings

Next we describe situations where although forcing with some poset $\mathbb{P}$ does destroy measurability, we can still "lift" an elementary embedding to a generic extension. More precisely, suppose that $j : V \to M$ is an elementary embedding with critical point κ, and $\mathbb{P}$ is a forcing notion in V. Let G be $\mathbb{P}$-generic. We ask the following:

Question: When can we extend j to an elementary $j' \supset j$, such that $j' : V[G] \to M[G^*]$, where G^* is $j(\mathbb{P})$-generic over M?

If $\mathbb{P}$ is such that we can do this, we say that we *lift* j to j', and in general j' will belong to some forcing extension of $V[G]$. Note that by the above theorem if the poset has small enough size, we can do that and actually get $j' \in V[G]$. The following lemma describes the general requirement for lifting embeddings.

Lemma 3. *Suppose that $j : V \to M$ is an elementary embedding with a critical point κ, and $\mathbb{P}$ is a forcing notion in V. Suppose also that G is a $\mathbb{P}$-generic filter over V, and G^* is a $j(\mathbb{P})$- generic filter over M such that j"$G \subset G^*$. Say that $G^* \in V[G][H]$, where $V[G][H]$ is some generic extension of $V[G]$. Then j can be lifted to an embedding $j' : V[G] \to M[G^*]$, where $j' \in V[G][H]$.*

Proof. For $a \in V[G]$, define $j'(a) = j(\dot{a})_{G^*}$, where $\dot{a}$ is some $\mathbb{P}$-name for a. Note that by elementarity $j(\dot{a})$ is a $j(\mathbb{P})$-name. We have to show that j' is well defined.

Suppose that τ and σ are two $\mathbb{P}$-names in V for an element $a \in V[G]$. More precisely, $\sigma_G = \tau_G = a$. Then there is some $p \in G$, such that $p \Vdash \sigma = \tau$. Then by elementarity $j(p) \Vdash j(\sigma) = j(\tau)$. Since $j(p) \in j$"$G \subset G^*$, we have that $M[G^*] \models j(\sigma)_{G^*} = j(\tau)_{G^*}$. Thus the map is well defined.

Next we show it is an embedding. Suppose that $V[G] \models \phi(\sigma^1_G, ..., \sigma^n_G)$. The by the forcing theorem, there is $p \in G$, such that $p \Vdash \phi(\sigma^1, ..., \sigma^n)$, so by elementarity of j, $j(p) \Vdash \phi(j(\sigma^1), ..., j(\sigma^n))$. Then, since $j(p) \in G^*$, by the forcing theorem applied to $j(\mathbb{P})$, $N[G^*] \models \phi(j(\sigma^1)_{G^*}, ..., j(\sigma^n)_{G^*})$. Since every $j(\sigma^i)_{G^*} = j(\sigma^i_G)$, that means that $N[G^*] \models \phi(j(\sigma^1_G), ..., j(\sigma^n_G))$. For the other direction, we use the same argument to show that if $V[G] \models \neg\phi(\sigma^1_G, ..., \sigma^n_G)$, then $N[G^*] \models \neg\phi(j(\sigma^1_G), ..., j(\sigma^n_G))$. □

Theorem 4. *Let $\mathbb{P} = Add(\omega, \kappa)$. Suppose that there is an elementary embedding $j : V \to N$ with critical point κ. Then for any G-$\mathbb{P}$-generic over V, there is H-$j(\mathbb{P})$-generic over N, such that we can extend j to $j' : V[G] \to N[H]$.*

Proof. First note that for every $p \in \mathbb{P}$, $j(p) = p$, and so j"$\mathbb{P} = \mathbb{P}$. Also, $j(\mathbb{P}) = Add(\omega, j(\kappa))$ and for every $p \in j(\mathbb{P})$, $p \restriction \kappa \times \omega \in \mathbb{P}$.

Let G be $\mathbb{P}$-generic over V. In $V[G]$, define the poset $\mathbb{A} = \{p \in j(\mathbb{P}) \mid p \restriction \kappa \times \omega \in G\}$, with the reverse inclusion ordering. Let H be $\mathbb{A}$-generic over $V[H]$.

Claim 5. *Let $p \in G$. Then $D := \{q \in \mathbb{A} \mid q \restriction \kappa \times \omega \leq p\}$ is a dense subset of $\mathbb{A}$.*

Proof. For any $q \in \mathbb{A}$, both $q \restriction \kappa \times \omega$ and p are in G, so we can find a common extension $r \in G$. Then $r \cup q \leq q$ and it is in D. □

It follows that for every $p \in G$, there is $q \in H$ with $q \restriction \kappa \times \omega \leq p$. By upward closure of H, we get that $p \in H$. I.e. j"$G \subset H$.

Now suppose that $D \in V$ is a dense subset of $j(\mathbb{P})$. For a condition $p \in j(\mathbb{P})$, denote $p_l := p \restriction \kappa \times \omega$ and $p^u := p \restriction [\kappa, j(\kappa) \times \omega$.

Claim 6. *For every $p \in j(\mathbb{P})$, $D_p = \{r \in \mathbb{P} \mid (\exists q \leq p) r \cup q^u \in D\}$ is a dense subset of $\mathbb{P}$.*

Proof. Let $p \in j(\mathbb{P})$ and $r \in \mathbb{P}$. By density of D, there is $q \leq r \cup p^u$ with $q \in D$. Then $q_l \leq r$ and $q_l \in D_p$.

□

In $V[G]$, define $D^* = \{p \in \mathbb{A} \mid (\exists p')(p' \in D, p \leq p')\}$.

Claim 7. *D^* is a dense subset of $\mathbb{A}$.*

Proof. Let $p \in \mathbb{A}$. By density of D_p, let $r \in D_p \cap G$. Let $q \leq p$ witness that $r \in D_p$. I.e. $q \leq p$ and $r \cup q^u \in D$. Now let $q' = r \cup q^u \cup p$. This is a well defined function since $q^u \leq p^u$, and both p_l and r are in G. Then, since $q'_l = r \cup p_l \in G$, we have that $q' \in \mathbb{A}$. Also since $q' \leq r \cup q^u \in D$, we get that $q' \in D^*$. □

So let $p \in H \cap D^*$. Then there is some $p' \in D$ with $p \leq p'$. By upward closure of H, $p' \in H$. We have shown that H is $j(\mathbb{P})$-generic over V (and so over N), such that $j"G \subset H$. By the previous lemma this means that we can extend the embedding to $j : V[G] \to N[H]$. □

Note that one of the key points in the proof is that $j(\mathbb{P})$ projects to $\mathbb{P}$. The next theorem generalizes that idea.

Theorem 8. *Suppose that there is an elementary embedding $j : V \to N$ with critical point κ, and $\mathbb{P}$ is a forcing notion such that $j(\mathbb{P})$ projects to $\mathbb{P}$. Then for any G-$\mathbb{P}$-generic over V, there is H-$j(\mathbb{P})$-generic over N, such that we can extend j to $j' : V[G] \to N[H]$.*

Proof. Let $\pi : \mathbb{P}^* \to \mathbb{P}$ be a projection. In $V[G]$, let $\mathbb{A} = \{p \in \mathbb{P}^* \mid \pi(p) \in G\}$. Let H be $\mathbb{A}$-generic over $V[G]$. Exactly as before we show that $j"G \subset H$ and H is $j(\mathbb{P})$-generic over V. The we can lift j. □

Suppose that $\pi : \mathbb{P}^* \to \mathbb{P}$ is a projection, and G is $\mathbb{P}$-generic. We will write $\mathbb{P}^*/G := \{p \in \mathbb{P}^* \mid \pi(p) \in G\}$. By the above if H is $\mathbb{P}^*/G$-generic over $V[G]$, then $V[G][H] = V[H]$, and we can write $\mathbb{P}^*$ is the two step iteration $\mathbb{P} * (\mathbb{P}^*/\dot{G})$.

Next we show that we can lift an elementary embedding after forcing with the Mitchell poset $\mathbb{M}$.

Theorem 9. *Let $j : V \to N$ be an elementary embedding with critical point κ. Then $j(\mathbb{M})$ projects to $\mathbb{M}$.*

Proof. By elementarity conditions in $j(\mathbb{M})$ are pairs (p, q), such that:

(1) $p \in j(\mathbb{P}) = Add(\omega, j(\kappa))$,
(2) $\text{dom}(q)$ is a countable subset of $j(\kappa)$, and for all $\alpha \in \text{dom}(q)$, $1_{Add(\omega,\alpha)} \Vdash$ "$q(\alpha) \in \dot{Add}(\omega_1, 1)$".

The ordering is the natural extension of the ordering for $\mathbb{M}$. Define $\pi : j(\mathbb{M}) \to \mathbb{M}$ by $\pi(p,q) = (p \restriction (\kappa \times \omega), q \restriction \kappa)$. We will show that this is a projection.

If $(p', q') \leq_{j(\mathbb{M})} (p,q)$, then:

- $p' \supset p$, so $p' \restriction (\kappa \times \omega) \leq_{\mathbb{P}} p \restriction (\kappa \times \omega)$,
- $\mathrm{dom}(q') \supset \mathrm{dom}(q)$, so $\mathrm{dom}(q' \restriction \kappa) \supset \mathrm{dom}(q \restriction \kappa)$, and
- for all $\alpha \in \mathrm{dom}(q)$, $1_{Add(\omega,\alpha)} \Vdash q'(\alpha) \leq q(\alpha)$, and in particular this is true for all $\alpha \in \mathrm{dom}(q \restriction \kappa)$.

So, $(p' \restriction (\kappa \times \omega), q' \restriction \kappa) \leq_{\mathbb{M}} (p \restriction (\kappa \times \omega), q \restriction \kappa)$; i.e. the map is order preserving.

For the second requirement of being a projection, suppose that $(p,q) \in j(\mathbb{M})$, $(r,s) \in \mathbb{M}$ and $(r,s) \leq \pi(p,q)$. Let $p' = p \cup r$. Since $r \leq p \restriction (\kappa \times \omega)$, this is a well defined function, and $p' \in j(\mathbb{P})$, such that $p' \leq_{j(\mathbb{P})} p$ and $p' \restriction (\kappa \times \omega) = r$.

Next we define q'. Set $\mathrm{dom}(q') = \mathrm{dom}(q) \cup \mathrm{dom}(s)$. For $\alpha \in \mathrm{dom}(q)$, of $\alpha < \kappa$, let $q'(\alpha) = s(\alpha)$. Otherwise, if $\alpha \geq \kappa$, let $q'(\alpha) = q(\alpha)$. Then $(p', q') \leq (p,q)$, and $\pi(p', q') = (r,s)$

□

Corollary 10. *If $j : V \to N$ is an elementary embedding with critical point κ, and G is $\mathbb{M}$ over V, then we can extend j to $j : V[G] \to N[G^*]$.*